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ABSTRACT

This document asserts that mathematics is becoming essential to many more disciplines. The paper is an attempt to describe the sorts of curricular changes which many mathematics educators are now recommending. The paper discusses: (1) mathematical power; (2) mathematics instruction; (3) new research directions; (4) mathematical thinking and learning; (5) rote learning; (6) the development of subject matter; (7) problem solving strategies; (8) metacognition and "executive control"; (9) the use of the calculator and computer technology; (10) the effects of calculators and computers; and (11) future directions of research. It concludes that the need for a structural scientific approach to curriculum and instructional development in mathematics has never been greater. A four-page list of references is provided. (TW)

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TOWARD A SCIENTIFIC PRACTICE OF MATHEMATICS EDUCATION

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The Need for Change

The nature of mathematics is changing rapidly: both the techniques of investigation and the areas of research interest. In addition, mathematics is becoming essential to many more disciplines. The explosive growth of new technologies has increased the number and variety of useful applications. Calculators and computers are increasing the need for mathematical knowledge by making previously qualitative disciplines (from literature to political science) more quantitative. Calculators are decreasing the need for computation and placing greater demands on analytical and thinking skills.

We have every reason to expect that the hectic pace of change in mathematics will continue through the foreseeable future. Thus, mathematics education must learn to adjust to a situation in which each new development will be superseded before it is widely implemented. Educational research will

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be part of a process of continuous development and renewal. Determining appropriate goals in this uncertain environment is a daunting task. Any specific skill we choose to teach, whether computational or high-level-cognitive, could become obsolete before the learner ever has need of it. It is a situation in which curriculum design deserves very careful thought and presents a challenge to even the most experienced teacher. Groups of students who in the past needed very little mathematics now must be taught to perform applications which they will need in their future work and which use techniques which have not yet been invented.

This chapter is an attempt to describe the sorts of curricular changes which many of those in mathematics education (though by no means all) of those in mathematics education are now recommending. There is wide recognition that more research needs to be done and thus that each innovation should be carefully monitored before it is widely implemented. After describing the general direction of change in mathematics education, I will consider relevant research and finally indicate some of the questions that ought to guide future investigations.

Changes that are needed

What are the key modifications needed in the K-12 mathematics curriculum? Some entail increased emphasis on traditional objectives such as the development of number sense and symbol sense. Other changes involve the introduction of material rarely found in the current curriculum, such as data analysis, graph theory and probability. Still others involve de-emphasis of topics whose aim has been to develop certain manipulative skills that are no longer very useful, such as long division and factoring trinomials.

But far more important than changing the content of the curriculum, is the need to encourage students to develop a spirit of inquiry, an intellectual curiosity and a sense of mathematical power.

Mathematical Power

Education in all subjects requires a balance between developing skills and knowledge, and the ability to deploy that knowledge. But computers are changing the equation and some skills are no longer prerequisites to further study. Once it was essential to stress penmanship before there could be a focus on what had been written; word processors have shifted the balance back to the content of the writing.

Calculators mean that we can shift the emphasis in K-12 mathematics away from skill development and toward mathematical power. This means the development of the abilities to:

- understand mathematical concepts and methods;
- discern mathematical relations;
- reason logically; and
- apply mathematical concepts, methods, and relations to solve a variety of nonroutine problems.

Students who achieve a considerable degree of mathematical power during their K-12 years will be able to use mathematics in their everyday lives or in a profession or vocation and they will be able to pursue further study of mathematics or other subjects that require mathematics. But computational power itself is not enough. Students must also learn to read and understand mathematical texts and to communicate to others orally and in writing results of their own mathematical investigations and problem-solving. The mathematics curriculum should provide support for the teaching of reading, writing and oral communication.

Calculator and computer technology should be used throughout the K-12 mathematics curriculum and new curriculum materials should be designed in the expectation of continuous change resulting from further scientific and technological developments.

Modern relevant applications should be a fundamental part of the curriculum to a much greater extent than at present.

"Applications" need not be constrained to "real world" problems. The significant criterion for the suitability of an application is whether it has the potential to engage the students; often this can be done with questions of purely mathematical interest such as "what is the largest prime number?".

Mathematics Instruction

Mathematical teaching must adapt to new realities. It will no longer be appropriate for most mathematics instruction to be in the traditional mode of teacher-presenting-material-to-a-class. Thinking mathematically is an active conception which requires more than listening. No single teaching method nor any single kind of learning experience can develop the varied kinds of mathematical abilities needed for mathematical power. The Cockroft Report (Cockroft, 1982) indicates some of the range of activities needed:

- Exposition by the teacher;
- Discussion between the teacher and the pupils
- Discussion between pupils themselves
- Practical work
- Consolidation and Practice of fundamental skills and routines;

- Problem solving, including the application of mathematics to everyday situations;
- Investigational work

The standard mode of presentation in which a teacher lectures to students may still be appropriate for the delivery of straightforward information, but more imaginative settings are needed for the development of problem-solving and reasoning skills, particularly in the context of using calculator and computer technology. For example, two formats which should be used often are:

- small group work where the class is divided into teams of, say, three to five students who work collaboratively on assigned problems (which might take anywhere from five minutes to two weeks to solve)
- true class discussion in which the teacher plays the role of moderator rather than leader.

In both of these formats, the teacher can be a catalyst who helps students learn to think for themselves rather than having the teacher act as a trainer whose role is to show the "right way" to do something. Both formats also allow the teacher to use technology interactively with students.

A useful metaphor is that of the teacher as a sort of intellectual coach. At various times, this will require the teacher to be:

- a role model who demonstrates not just the "right way" but also the false starts and the higher-order thinking skills that lead to the resolution of problems;
- a consultant helping individuals, small groups, or the whole class to decide if their work is keeping to the subject and making reasonable progress;

- a moderator who poses questions for the class (or individuals or groups) to consider, but leaves most of the decision-making to the class;
- an interlocutor coaching the students during class presentations, encouraging them to reflect on their activities and to explore mathematics on their own, challenging them to make sure that what they are doing is reasonable and purposeful, and ensuring that students can defend their conclusions.

Research Directions

Research on teaching for higher-order thinking, (Peterson, in press), lends support to the notion that instruction needs to change from the traditional, teacher-presenting-material-to-the-class mode to a less structured, indirect style of teaching. Because the development of higher level thinking in mathematics has been shown to depend on autonomous, independent learning behavior, teachers should encourage self-reliance. One type of indirect instruction that has often proved to be effective is small group cooperative learning (Lochhead, 1985; Peterson, in press; Shavelson, in press). Noddings (in press) pointed out that among the several benefits of small group learning is that small groups allow consultation, a heuristic that we all use when we meet up with difficulties. Cognitive research in other content areas (Brown & Campione, in press), using a reciprocal teaching model that includes children taking turns playing teacher and posing questions, summarizing, clarifying and predicting, has been effective in producing self-monitoring. Reciprocal teaching is based on the premise that the opportunity to communally construct meanings produces an internalization of the process of meaning construction (Resnick, in press).

Mathematical Thinking and Learning: Findings and Implications.

We have called for new modes of teaching which stress the active role students must play in the construction of their own concepts. There is now wide agreement among researchers (Resnick (1983); and Linn (1986)) of the need to pay careful attention to student constructed knowledge (Piaget, 1954). For example, Resnick (1976), Carpenter, Moser, & Romberg (1982), and Steffe et. al. (1983) have shown that students invent "counting on from larger" for themselves (when adding two numbers, say 3+6, the answer is found by counting 7,8,9). It is now clear that children come to school with a rich body of knowledge about the world around them, including well developed informal systems of mathematics (Ginsburg, 1977). Education fails when children are treated as "blank slates" or "empty jugs", ignoring the fact that they have a great deal of mathematical knowledge -- some of which surpasses, and some of which may contradict, what they are being taught in school (Clement, 1977 and Erlwanger, 1974; Ginsburg, 1977; Gelman & Gallistal, 1978) that can be exploited in children.

Rote Learning

Probably the most controversial recommendations concerns reducing the emphasis on rote learned procedures and on algorithms used for extensive paper and pencil calculations. There is extensive evidence that algorithms, in themselves, do not aid conceptual understanding. The literature on arithmetic "bugs" (see, e. g., Brown & Burton, 1978; Maurer, 1987) documents this point. Research reveals that in some 40% of the mistakes students make in subtraction one can describe the flawed procedure that produces the student's answers, and that predict incorrect answers students will produce on similar problems. Such consistent but mistaken procedures have a natural origin, in the invention of the student. Most of the student "bugs" -- so named because, like bugs in computer programs, they produce consistent incorrect answers -- can be explained as intelligent attempts to "patch" rote-learned algorithms that are poorly understood.

Even when correctly learned, purely "procedural knowledge" -- the ability to implement mathematical algorithms, but without the underlying conceptual structures -- can be extremely fragile. Clement, et al (1979) have shown that even a solid procedural knowledge of algebra, such as is held by university level engineering students, does not in most cases (over 80%) imply an ability to interpret the meaning of algebraic symbols. One can minimize the fragility of knowledge structures by teaching mathematical concepts in a fashion that stresses the underlying conceptual models (e.g. Carpenter, Moser, & Romberg, 1982; Davis, 1984; Hiebert, 1986; Romberg & Carpenter, 1986). In sum, we now know children are active interpreters of the world around them, including the mathematical aspects of that world (In Piaget's words, "to understand is to invent." (1973)). This suggests that topics in school should be arranged to exploit intuitions and informal numerical notions students bring with them to school. Second, it indicates that predominant teaching methods must be revised to adapt to the notion of child as <u>interpreter</u> (and constructor of possibly wrong theories) as opposed to child as absorber.

Development of Subject Matter

Researchers have only just begun to construct a detailed map of the phases children can go through as they gradually build up their understanding of number and arithmetic (Steffe et al, 1983.) Even at the early ages, the picture is quite complex. As students develop, it is most effective to engage them in meaningful, complex activities focusing on conceptual issues, rather than establishing all the building blocks firmly before going on to the next "level". (Collins, Brown & Newman, 1987; Hatano, 1982; Romberg & Carpenter, 1980). In certain cases, the order of presentation is critical; early introduction of some topics can be very damaging. Wearne and Hiebert (1987) have shown that if calculational algorithms are memorized before the underlying structure is understood, then "it may be difficult for semantic information to penetrate routinized rules" (p. 26, Wearne & Hiebert, 1987). In short, students who learn to calculate too early may find it more difficult to reach an understanding of the material than students who have had no such experience. But this is not always the case. In fact, Steffe et al (1983) showed that in some cases, memorized routinized rules must precede understanding. Children must be able to recite the number words in order (one, two, three...) before they can develop a concept of counting or number. In contrast, there is some evidence[1] to suggest that calculation algorithms involving fractions, decimal long division, and possibly multiplication are introduced far too soon in the current curriculum. The challenge for curriculum development (and research) is to determine when routinized rules should come first, and when they should not. This is an area where far more research needs to be done.

1 Evidence supporting the delayed introduction of fraction and decimal calculational algorithms comes mainly in the form of the large number of students who never learn these topics. The National Assessments indicate that a very high percentage of high school students worldwide never master these topics. This is what one would expect in a case where routinized skills are blocking semantic learning. The experience of Benazet (1935), who delayed such instruction until after 6th grade, indicates that such a postponement can be very helpful.

Problem Solving Strategies

There is an extensive body of literature (see, e.g., Charles & Silver, in press; Kulik, 1980; Mason, Burton & Stacey, 1982; Schoenfeld, 1985; Silver, 1983) indicating that problem solving strategies can be taught, and suggesting various ways to do so. The main warning from the research literature is that one should be careful not to trivialize problem solving strategies, teaching a collection of isolated tricks (e.g. "of" means multiply, or cross-cancelling factors). Problem solving strategies, in the spirit of Polya (1945), are subtle. Important strategies such as "look for a pattern by plugging in values for n = 1,2,3,4...." cannot be taught effectively, apart from situational cues which indicate when it is appropriate to apply them.

Metacognition and "Executive Control"

An important aspect of problem solving competence is metacognition - the ability to know when and why to use a procedure. There is ample evidence (Collins, Brown, & Newman, in press; Brown, Ferrara & Campione, 1983; Schoenfeld, 1985; Silver 1985) that students who "know" more than enough domain-specific subject matter fail to solve problems because they do not use their knowledge wisely. They may jump into problems, doggedly pursuing a particular ill-chosen approach to the exclusion of anything else; they may raise profitable alternatives, but fail to pursue them; they may get sidetracked into focusing on trivia while ignoring the "big picture."

Research indicates that such "executive" skills can be learned, resulting in significant improvements in problem solving performance. Effects can be obtained with interventions as simple as holding class discussions that focus on executive behaviors, and by explicitly and frequently posing questions such as:

What are you doing?

Why are you doing it?

How will it help you?

(Schoenfeld, 1985; Collins, Brown, & Newman, in press.)

Beliefs: Getting a sense of what mathematics is about.

On the Third National Assessment of Educational Progress (Carpenter et. al., 1983), a stratified nationwide sample of 45,000 students worked the following problem:

> *An army bus holds 36 soldiers. If 1,128 soldiers are being bussed to their training site, how many buses are needed?*

Roughly 29% of the students who worked the problem wrote that the number of buses needed is "31 remainder 12," but only 23% gave the correct answer to the problem. Approximately 70% of the students who took the examination performed the right operation (1,128 divided by 36 yields "31 remainder 12"). Then, however, fewer than 1/3 of those students wrote 32 buses. How can students say that the number of buses includes a remainder?

For most students, the "school mathematics mode" includes a habit of problem solving without sense-making: one learns to read the problem, extract the relevant numbers and the operation to be used, perform the operation, and write down the result (Lave, in press; Reusser, in press; Schoenfeld, in press-b). Consider, for example the following nonsense problems:

> *There are 26 sheep and 10 goats on a ship.*
> *How old is the captain?*
>
> *There are 125 sheep and 5 dogs in a flock.*
> *How old is the shepherd?*

Reusser (In press) reports that, asked to "solve" such problems, three school children in four will produce a numerical answer. There are similar data for French school children, and the NAEP data speak for themselves. In a discussion of these and similar problems, Reusser suggests that the students work the problems compliantly, and without asking that they make sense, because the students have already "learned' that school math problems do not necessarily make sense. In the context of the mathematics classroom, the expectation is that problems have an answer (why else would the teacher pose them?) and that some reasonable combination of the numbers in the problem (usually using the most recent mathematical procedure studied) will yield it. Students learn to act in the way described in the previous paragraph.

Students constantly strive to make sense of the rules that govern the world around them, including the world of their mathematics classrooms. If the classroom patterns are perceived to be arbitrary and the mathematical operations meaningless -- no matter how well "mastered" as procedures - students will emerge from the classroom with a sense of mathematics being arbitrary, useless, and meaningless. Consider the example (Reusser, in press) of a student working the shepherd problem. This solution, produced by a student solving the problem out loud, is all too typical:

> *"125 + 5 = 130... this is too big, and 125 - 5 = 120 is still too big... while 125/5 = 25. That works. I think the shepherd is 25 years old."*

In short, the "classroom culture" in which students learn mathematics shapes their developing understanding of the nature of mathematics -- which, in turn, shapes how sensibly students will use the mathematics they have learned. Research indicates (Fawcett, 1938; Lampert, 1987; Lave, in press; Mason, Burton, & Stacey, 1982; Schoenfeld, in press-b) that it is indeed possible to create classroom environments that are, in essence, cultures of sense-making -- and from which students emerge with an understanding of mathematics as a discipline which helps to make sense of things. The goal of teaching sense-making *via* mathematics should be a central concern of our curricular efforts.

Use of Calculator and Computer Technology

At the turn of the century a social planner might have expressed concern over the consequences of widespread access to the automobile. But in the last analysis research evidence pro or con the automobile would have been irrelevant and ignored. The type of study that might have made sense would have assumed the car and considered how best to use it. We believe the same is true of computers and calculators. Like it or not, they are a fact of life.

Computers have already changed the face of mathematics. New fields of inquiry, such as fractal geometry, depend in large part on the computer for their very existence. Much of modern mathematics is inaccessible and inexplicable without access to computers.

But it is still reasonable to ask whether computers and calculators in the curriculum may not pose some serious dangers. In particular, should they be introduced early before students have mastered the basics?

Effects of Calculators

The effects of calculators in school mathematics have been studied in over 100 formal investigations during the past 15 years. Those studies have tested the impact of a variety of kinds of calculator use--from limited access in carefully selected situations to access for all aspects of mathematics instruction and testing. There have been two major summaries of reported research on calculator usage (Sudyam, 1982; Hembree and Dessart, 1986). In almost every reported study, the performance of groups using calculators equalled or exceeded that of control groups denied calculator use.

The recent Hembree and Dessart meta-analysis of 79 calculator studies sorted out the effects of calculator use on six dimensions of attitude toward mathematics as well as on acquisition, retention, and transfer of computational skill, conceptual understanding, and problem solving ability. The analysis led them to conclude:

> 1. Students who use calculators in concert with traditional instruction maintain their paper-and-pencil skills without apparent harm. Indeed, a use of calculators can improve the average student's basic skills with paper and pencil, both in basic operations and in problem solving.
>
> 2. One study reported that sustained calculator use by average students in Grade 4 appears counterproductive with regard to basic skills.
>
> 3. The use of calculators in testing produces much higher achievement scores than paper-and-pencil efforts, both in basic operations and in problem solving. This statement applies across all grades and ability levels. In particular, it applies for low- and high-ability students in problem solving. The overall better performance in problem solving appears to be a result of improved computation and process selection.
>
> 4. Students using calculators possess a better attitude toward mathematics and an especially better self-concept in mathematics than noncalculator students. This statement applies across all grades and ability levels.
>
> 5. Studies with special curricula indicate that materials and methods can be developed for enhancing student achievement through instruction oriented toward the calculator. However, such special instruction has been relatively unexamined by research.
>
> *(Source: Hembree and Dessart, 1986, pp. 96-97)*

These findings speak directly to a number of common concerns about the potential impact of widespread calculator use in school mathematics. For those who believe that some measure of skill in traditional arithmetic algorithms will continue to be important for most students to acquire, the research suggests that access to calculators in a well-planned program of instruction is not likely to obstruct achievement of those skills. In fact, it might well enhance acquisition of traditional skills. More optimistically, it appears that when students have access to calculators for learning and achievement testing, they perform at significantly higher levels on both computation and problem solving measures. In particular, students using calculators seem better able to focus on correct process analysis of problem situations.

Effects of Computers

The earliest educational use of computers was primarily to deliver computer assisted instruction, often in a programmed learning style of instruction, and most frequently, for drill or rote skills. Several reviews of research on effectiveness of CAI (Kulik, et al, 1986) have concluded that it is generally very effective, giving better achievement in shorter time than traditional instruction.

Lately, principles of artificial intelligence have been applied to the design of sophisticated tutors for algebra, geometry, and calculus. The designers suggest that the use of such tutors can yield dramatic increases in student achievement. However, no data is available about the use of such tutors in realistic classroom settings.

There are several kinds of computer-based strategies for giving students impressive new learning tools and exploratory environments (Hansen, 1984; Pea, 1987, Schoenfeld, in press a). Best known is Logo and its turtle graphics feature to teach students concepts of geometry, algebra, and general higher order thinking skills (Papert, 1980). Research findings have failed to confirm the strongest claims that Logo develops high level general reasoning abilities. But a variety of studies have found positive effects on more specific instructional goals (Campbell, 1987), and thousands of classroom teachers have been convinced by first-hand experience that LOGO is a powerful instructional tool.

A somewhat different sort of computer-based exploratory tool has been provided by the Geometric Supposer (Schwartz & Yerushalmy, 1987), and by the Geodraw software developed at Wicat (Bell, 1987). In each, the idea is to give students open but guided environments for exploring the results of geometric constructions. A comparable setting for algebraic exploration is provided by the Green Globs software of Dugdale (1982). There is little formal research describing the effects of these learning and teaching tools. Yerushalmy, Chazan, and Gordon (1987) provide evidence suggesting that students using the SUPPOSER may perform as well or better than control students on traditional geometry criteria, while at the same time learning other objectives as well.

There have been some specific research studies investigating the effects of computer graphics on student understanding of mathematical concepts like function (Rhoads, 1986; Schoenfeld, 1988). The curriculum development work of Demana, Leitzel, Osborne, and Waits at Ohio State University has taken particular advantage of spreadsheet-like software to develop students' numerical intuition about variables and equations in pre-algebra and pre-calculus mathematics. In each case, the computer seems clearly to enhance student interest and understanding of important ideas.

Most studies have focused, in one way or another, on finding better ways to reach traditional goals. There have been some noteworthy exceptions to that pattern--studies that explore daring departures from conventional curriculum priorities. Both Lesh (1986) and Heid and Kunkle (1988) tested the effects of algebra instruction in which students used symbol manipulation software to perform routine tasks like solving equations. Each found that students who were freed from the traditional symbolic procedural aspects of problem solving became much more adept at the important problem formulation and interpretation phases. In two similar studies of computer-aided calculus, Heid (1988) and Palmiter (1986) found that students who learned the subject with aid of computer tool software developed much deeper understanding of fundamental concepts than did students in traditional skill-oriented courses. Heid also found that her students picked up needed procedural knowledge in a short time period following the careful conceptual background, and Palmiter found that her students acquired their understanding much more quickly than students in conventional courses.

Each of these studies addresses the fundamental question of technology applied to mathematics curriculum: what are the essential interactions of conceptual and procedural knowledge and problem solving ability? If we diminish attention to the traditional procedural skill agendas in various branches of mathematics, will something essential to problem solving or conceptual learning be inadvertently lost? A fair test of this question involves extensive and radical curriculum development and field testing with attendant risks for students who study the new curricula. Not surprisingly, the only work of this type has been in limited numbers of classes and situations. At the University of Maryland, a computer-intensive elementary algebra program has been developed to explore the feasibility of teaching fundamental concepts and problem solving abilities while using technology to perform nearly all of the traditional symbol manipulation skills. Preliminary evidence indicates that students can approach algebra as the study of functions and their application as models of quantitative interactions, that they can become flexible and effective algebraic problem solvers, and that the rich conceptual background of understanding about variables, functions, equations, and inequalities they acquire provides a strong foundation for the more abstract task of learning appropriate symbol manipulation skills later (Heid, 1988).

The research cited above indicates that access to computers and calculators need not hinder attainment of traditional curricular objectives, and that it may substantially advance it. Unfortunately, there is currently no consensus on how to investigate possible dramatically new effects such as the improvement of higher order thinking skills. A series of articles in Educational Researcher (Papert (1987); Pea (1987); Walker (1987); and Becker (1987)) illustrates the wide diversity of opinion on this topic. A key concern is the extent to which the development of powerful reasoning can be inferred from written test performance or within the limited time spans of most research studies.

From the few attempts that have been made to measure massive changes in reasoning power it is possible to conclude that such advancements cannot come from trivial technological fixes. It has often been propcsed that the availability of computers would, more or less in itself, produce significant improvements in mathematical thinking. Repeated attempts to document such change has yet to reveal a lasting effect: e.g. Pea's study of the effect of LOGO on planning, Soloway's investigation of the impact of PASCAL on understanding of algebraic syntax and Perkin's research concerning the cognitive impact of learning programming meta-principles in BASIC. While these results do not imply that computers cannot be used to improve mathematical thinking, they suggest that simplistic approaches are not likely to work. Thus teachers and curriculum designers probably will have a significant role to play in education, far into the computer age. The computer will remain a tool for teachers and students to use, thus integration of technology into "Classroom Ecology" must remain a high priority research item.

Future Directions for Research

Order of Learning

Many topics can be taught earlier than they have been, others ought to be taught later. Since it is impossible to forecast the full implications of such changes, they should first be implemented in a research environment. Such studies must go beyond evaluating student progress in terms of superficial measures, they must examine deep conceptual understanding as well as the long-term effects of gaining or not gaining such understandings at a certain stage.

Levels of Learning

It seems clear that the new representations and computational aids made possible through computer technology now allow us to teach some concepts much earlier than they had been previously. Research needs to be done on the degree to which students really understand advanced concepts when they are introduced early. The goal of such research should be to find a suitable sequencing and pace for the introduction of new topics.

Modes of Learning

We have discussed the growing evidence supporting the need for new modes of instruction. The theoretical basis for many of the proposed new techniques is recognition of the active role students must take in constructing their own knowledge. Studies are now needed that can determine the long term effects of early exposure to various different modes of mathematical instruction on mathematical competence and on facility at learning new mathematical concepts. In evaluating instructional techniques it is important to avoid criteria which value maximizing current performance without regard to the extended impact.

The Roles of Arithmetic and Algebraic Manipulative Skills

With-in the next decade hand-held calculators capable of performing symbolic manipulations (e.g. solving linear and quadratic equations in one variable, pairs of linear equations in two variables, etc.) will become widely available. Just as the current generation of calculators allows for the rearrangement of arithmetic topics (postponing some paper and pencil rote skills until appropriate) the next generation of calculators will allow much more flexibility in the order of advanced topics: allowing students, for example, to do much less algebraic symbol manipulations. We must, therefore, formulate a new set of fundamental manipulative skills, and determine when each of these skills should be introduced.

The Relation Between Drill-and-Practice and Understanding Mathematics

The questions of how much paper-and-pencil arithmetic to teach to elementary school children and how much symbol manipulation proficiency is desirable for secondary school students depends crucially on the correlation of such skills with achieving an understanding of the underlying mathematics.

Once the development of such skills could be justified on social and economic grounds. But the rapid advance of calculator and computer technology has undermined such a justification. The question of what, if any, level of manipulative skill is necessary in order to be able to understand - and thus, use - mathematics in a problem-solving context is a very difficult one on which research is badly needed.

Evaluations of the Effects of Entire Curriculae: the "Transfer Problem"

We need research which can help us to learn more about the ways in which mathematical experiences shape people's understanding of mathematics, understandings which often mitigate against the use of mathematics in real world situations. We need to study ways of developing curriculae that help to solve the "Transfer Problem": that is to provide students with the sort of background that will encourage, rather than discourage, their ability to apply what they have learned in out of school contexts.

Instructional Uses of Technology

The "information explosion" has just begun to result in tools that can have significant impact on the instructional process. What kinds of mathematical comprehension can these new tools foster (are there negative side-effects) and how do they fit within the context of schooling? Critical here is the question of access. If home computers become a significant part of the instructional process, what happens to those who do not have such tools?

Summary

The need for a structural scientific approach to curriculum and instructional development in mathematics has never been greater, and it is increasing at a very rapid rate. Sooner or later we may be forced to take it seriously.

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